Some Properties of The Isomorphic Fuzzy Subgroups

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Abstract

The purpose of this paper is to give some properties of the isomorphic fuzzy subgroups.

Keyword: Fuzzy groups, isomorphic fuzzy subgroups, primary fuzzy subgroup, level subgroup.

1 Introduction

The concept of fuzzy sets was first introduced by Zadeh in 1965 [6]. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld in 1971 [4]. In 1975 Negoita and Ralescu [3], considered generalization of Rosenfeld’s definition in which the unit interval \( I = [0, 1] \) was replaced by an appropriate lattice structure. In 1979 Anthony and Sherwood [1] redefined a fuzzy subgroup of a group using the concept of triangular norm. In fact many basic properties in group theory are found to be carried over to fuzzy groups. This paper continues that study by looking at isomorphic fuzzy subgroups.

The second section of this paper gives some conventions, definitions and theorems which are essential later in the paper. Some of the definitions can be found elsewhere, but we include them to help the reader. Our main is to give some properties of the isomorphic fuzzy subgroups in section three.

2 Preliminaries

Let us first recall some concepts related isomorphic fuzzy subgroups.

**Definition 2.1** Let \( \mu \) be a fuzzy subset of a group \( G \). Then \( \mu \) is called a fuzzy subgroup of \( G \) under a t-norm \( T \) (T fuzzy subgroup ) iff for all \( x, y \) in \( G \)

1. \( \mu(xy) \geq T(\mu(x), \mu(y)) \)
2. \( \mu(e) = 1 \), where \( e \) is the identity of \( G \).
3. \( A(x^{-1}) \geq A(x) \),
where the product of $x$ and $y$ is denoted by $xy$ and the inverse of $x$ by $x^{-1}$.

Note: $\mu$ is called a Min-fuzzy subgroup if $\mu$ satisfies conditions (1) and (3) only by replacing $T$ with Min.

**Definition 2.2** Let $\mu$ be a fuzzy subset of a group $G$. Then the subset \{ $x \in G, \mu(x) \geq t$ \}, $t \in [0,1]$, of $G$ is called a t-level subset of $G$ under $\mu$ and it is denoted by $\mu_t$.

**Definition 2.3** For $i = 1$ and 2, let $\mu_i$ be fuzzy subgroup of $G_i$. If there is a group isomorphism $\phi$ from $G_1$ onto $G_2$ such that $\mu_1 = \mu_2 \circ \phi$, then we say $\mu_1$ and $\mu_2$ are isomorphic.

**Definition 2.4** Let $\mu$ be a $T$-fuzzy subgroup of $G$ and $x \in G$. Then the least positive integer $n$ satisfying the condition $\mu(x^n) = 1$ is called the the fuzzy order of $x$ with respect to $\mu$ and we use the notation $\mu \bullet (x) = n$. If $n$ does not exist, we say $x$ of infinitive fuzzy order with respect to $\mu$ and write $\mu \bullet (x) = \infty$.

**Definition 2.5** Let $\mu$ be a $T$-fuzzy subgroup of $G$. Then the least common multiple of the fuzzy orders of the elements of $G$ with respect to $\mu$ is called the order of the fuzzy subgroup $\mu$ and it is denoted by $|\mu|_F$. If it does not exist then $|\mu|_F = \infty$.

**Theorem 2.6** Let $\mu$ be a $T$-fuzzy subgroup of $G$. Then,

1. If $\mu(x^r) = 1$, then $\mu \bullet (x)|r$, and
2. If $\mu \bullet (x) < \infty$, then $\mu \bullet (x)|\bigcirc (x)$, where $\bigcirc (x)$ is the order of $x$.

**Definition 2.7** Let $\mu$ be a $T$-fuzzy subgroup of $G$ and $p$ is a prime. Then $\mu$ is called primary fuzzy subgroup of $G$ if for every $x$ in $G$ there exists a natural number $r$ such that $\mu \bullet (x) = p^r$.

**Theorem 2.8** If $\phi$ is an isomorphism from $G_1$ onto $G_2$ and $\mu_2$ is an fuzzy subgroup of $G_2$ under $T$, then $\mu_1 = \mu_2 \circ \phi$ is an fuzzy subgroup of $G_1$ under $T$.

**Definition 2.9** For each $i = 1, 2, \ldots, n$, let $G_i$ be a fuzzy subgroup under a minimum operation in a group $X_i$. The membership function of the product $G = G_1 \times G_2 \times \cdots \times G_n$ in $X = X_1 \times X_2 \times \cdots \times X_n$ is defined by

$$(G_1 \times G_2 \times \cdots \times G_n)(x_1, x_2, \ldots, x_n) = \min(G_1(x_1), G_2(x_2), \ldots, G_n(x_n)).$$
3 Isomorphic fuzzy subgroups

Theorem 3.1 If φ is an isomorphism from $G_1$ onto $G_2$ and $\mu_1$ is a fuzzy subgroup of $G_1$ and $\mu_1 = \mu_2 \circ \phi$, then $\mu_2$ is a fuzzy subgroup of $G_2$.

Proof Since φ is an isomorphism from $G_1$ onto $G_2$, for all $x, y \in G_2$, there are $a, b \in G_1$ such that $\phi(a) = x$ and $\phi(b) = y$. Using definition of fuzzy subgroup and $\mu_1$ is a fuzzy subgroup, we have that

$$
\mu_2(xy) = \mu_2(\phi(a)\phi(b)) = \mu_2(\phi(ab))
= (\mu_2 \circ \phi)(ab) = \mu_1(ab)
\geq \min\{\mu_1(a), \mu_1(b)\}
= \min\{(\mu_2 \circ \phi)(a), (\mu_2 \circ \phi)(b)\}
= \min\{\mu_2(\phi(a)), \mu_2(\phi(b))\}
= \min\{\mu_2(x), \mu_2(y)\}
$$

and

$$
\mu_2(x^{-1}) = \mu_2(\phi(a)) = (\mu_2 \circ \phi)(a)
= \mu_1(a) = \mu_1(a^{-1}) = (\mu_2 \circ \phi)(a^{-1})
= \mu_2(\phi(a^{-1})) = \mu_2(x)
$$

Theorem 3.2 The product of isomorphic fuzzy subgroups are isomorphic.

Proof Let $A_i$ and $B_i$ be isomorphic fuzzy subgroup of $G_i$ and $H_i$, respectively and $\phi : G = G_1 \times \cdots \times G_n \rightarrow H = H_1 \times \cdots \times H_n$ an isomorphism. Then for all $x_i \in G_i$ we get that

$$(A_1 \times \cdots \times A_n)(x_1, \ldots, x_n) = \min((B_1 \circ \phi)(x_1), \ldots, (B_n \circ \phi)(x_n))
= \min(B_1(\phi(x_1)), \ldots, B_n(\phi(x_n)))
= (B_1 \times \cdots \times B_n)(\phi(x_1), \ldots, \phi(x_n))
= ((B_1 \times \cdots \times B_n) \circ \phi)(x_1, \ldots, x_n)$$

Proposition 3.3 Let $G_1$ and $G_2$ be groups, $\phi : G_1 \rightarrow G_2$ an isomorphism, $\mu_1$ and $\mu_2$ fuzzy subgroups of $G_1$ and $G_2$, respectively. $\mu_1$ is isomorphic to $\mu_2$ iff for all $x \in \text{Ker}\phi$, $\mu_1(x) = 1$.

Proof Let $\mu_1$ and $\mu_2$ be isomorphic fuzzy subgroups. For all $x \in \text{Ker}\phi$,

$$
\mu_1(x) = (\mu_2 \circ \phi)(x) = \mu_2(e) = 1.
$$

Conversely, suppose for all $x \in \text{Ker}\phi$, $\mu_1(x) = 1$.

$$
\mu_1(x) = 1 \geq \mu_2(\phi(x)) = (\mu_2 \circ \phi)(x),
$$
that is $\mu_2 \circ \phi \subseteq \mu_1$. Similarly,

$$(\mu_2 \circ \phi)(x) = \mu_2(\phi(x)) = \mu_2(e) = 1 \geq \mu_1(x)$$

So we have

$$\mu_1 = \mu_2 \circ \phi.$$  

**Proposition 3.4** Let $G_1$ and $G_2$ be groups, $\phi : G_1 \to G_2$ an isomorphism, $\mu_1$ and $\mu_2$ $T$-fuzzy subgroups of $G_1$ and $G_2$, respectively. Then $\mu_1 \cdot (x) = \mu_2 \cdot (\phi(x))$.

**Proof** Let $\mu_1 \cdot (x) = n$ and $\mu_2 \cdot (\phi(x)) = m$. Then $\mu_1(x^n) = 1$. Since $\mu_1$ and $\mu_2$ are isomorphic, we get

$$\mu_1(x^n) = (\mu_2 \circ \phi)(x^n) = \mu_2((\phi(x))^n) = 1.$$  

By Theorem 2.5, $m|n$. On the other hand,

$$\mu_2((\phi(x))^m) = (\mu_2 \circ \phi)(x^m) = \mu_1(x^m) = 1$$

and hence $n|m$. Therefore $n = m$.

**Proposition 3.5** Let $G_1$ and $G_2$ be groups, $\phi : G_1 \to G_2$ an isomorphism, $\mu_1$ and $\mu_2$ $T$-fuzzy subgroups of $G_1$ and $G_2$, respectively. If $\mu_1$ is a primary fuzzy subgroup of $G_1$, then $\mu_2$ is a primary fuzzy subgroup of $G_2$.

**Proof** If $\mu_1$ is primary fuzzy subgroup of $G_1$, then for every $x$ in $G_1$, there exists a natural number $r$ such that $\mu_1 \cdot (x) = p^r$. Hence $\mu_1(x^r) = 1$. Since $\mu_1$ and $\mu_2$ are isomorphic fuzzy subgroups, there exists $\phi$ isomorphism between $G_1$ and $G_2$ such that $\mu_1 = \mu_2 \circ \phi$. That is

$$\mu_1(x^r) = (\mu_2 \circ \phi)(x^r) = \mu_2((\phi(x))^r) = 1.$$  

From Theorem 2.5, we can write $\mu_2 \cdot (\phi(x))|p^r$. In here, there are two cases. Either for every $\phi(x) \in G_2$, $\mu_2 \cdot (\phi(x)) = 1$ or $\mu_2 \cdot (\phi(x)) = p^s$ where $s \leq r$ and $r, s$ are natural numbers. If $\mu_2 \cdot (\phi(x)) = 1$ then we get $\mu_2((\phi(x)) = 1$ that is

$$(\mu_2 \circ \phi)(x) = \mu_1(x) = 1.$$  

But it is not true. Thus $\mu_2 \cdot (\phi(x)) = p^s$. So $\mu_2$ is a primary fuzzy subgroup of $G_2$.

**Proposition 3.6** If $\mu_1$ and $\mu_2$ are isomorphic primary fuzzy subgroups, then $|\mu_1|_F = |\mu_2|_F$.

**Proof** From Proposition 3.2 for all $x \in G_1$, $\mu_1 \cdot (x) = \mu_2 \cdot (\phi(x))$. Using Definition of fuzzy subgroup, we can easily see that $|\mu_1|_F = |\mu_2|_F$. 

Proposition 3.7 If $\mu_1$ and $\mu_2$ are isomorphic primary fuzzy subgroups, then $\mu_{1t}$ and $\mu_{2t}$ are t-level subgroups of $\mu_1$ and $\mu_2$, respectively, are isomorphic.

Proof For all $x \in \mu_{1t}$,
$$\mu_1(x) = (\mu_2 \circ \phi)(x) = \mu_2(\phi(x)) \geq t,$$
that is $\phi(x) \in \mu_{2t}$ on the other hand for all $\phi(x) \in \mu_{2t}$,
$$\mu_2(\phi(x)) = (\mu_2 \circ \phi)(x) = \mu_1(x) \geq t.$$
Thus $x \in \mu_{1t}$. Therefore, there is one-to-one correspondence among elements of the $\mu_{1t}$ and $\mu_{2t}$. Since $\phi$ is a isomorphism, $\mu_{1t}$ and $\mu_{2t}$ are isomorphic subgroups.

Theorem 3.8 Let $A$ and $B$ be fuzzy subgroups of finite groups $G_1$ and $G_2$, respectively. If $A$ and $B$ are isomorphic, then $Im(A) = Im(B)$.

Proof Let $t \in Im(A)$. Since $A$ and $B$ are isomorphic, there exist $\phi : G_1 \rightarrow G_2$ isomorphism such that $A(x) = (B \circ \phi)(x)$. Then $A(x) = B(\phi(x)) = t$ and $t \in Im(B)$. On the other hand, let $t \in Im(B)$. It is clear that $B(y) = B(\phi(x)) = (B \circ \phi)(x) = A(x) = t$. That is $t \in Im(A)$. This complete the proof.

Theorem 3.9 Let $A$ and $B$ be isomorphic fuzzy subgroups of finite groups $G$ and $H$, respectively. If level subgroups of $A$ has a chain as
$$A_{t_0} < A_{t_1} < \cdots < A_{t_r} = G$$
for the numbers $t_0, t_1, \ldots, t_r$ which belong to $Im(A)$ with $t_0 > t_1 > \cdots > t_r$, then level subgroups of $B$ has a chain as
$$B_{t_0} < B_{t_1} < \cdots < B_{t_s} = H$$
with same length.

Proof Since $A$ is isomorphic to $B$, for all $i = 1, 2, \ldots, n$, there is a one to one corresponding between level subgroup $A_{t_i}$, and level subgroup $B_{t_i}$. Really
$$A(x) = (B \circ \phi)(x) = B(\phi(x)) = B(y) \geq t_i.$$ 
Because each level subset is a subgroup of $H$ in the usual sense, let we prove that $B_{t_i}$ is a subset of $B_{t_{i+1}}$. For all $y \in B_{t_i}$, $B(y) \geq t$. We get
$$B(\phi(x)) = (B \circ \phi)(x) = A(x) \geq t_i.$$
because of being isomorphic of $A$ and $B$. As $A_{t_i} < A_{t_{i+1}}$, $x \in A_{t_{i+1}}$. So

$$A(x) = (B \circ \phi)(x) = B(\phi(x)) = B(y) \geq t_{i+1}$$

and

$$y \in B_{t_{i+1}}.$$ 

Therefore $B_{t_i}$ is subset of $B_{t_{i+1}}$. From Theorem 3.8, $Im(A) = Im(B)$ so level subgroups of $A$ and $B$ has a chain with same length.

References


