EQUIVALENCE OF ONE-STEP, TWO-STEP AND THREE-STEP ITERATIVE SCHEMES FOR ZAMFIRESCU OPERATOR

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ABSTRACT

In this paper, we have shown that the convergence of one-step, two-step and three-step iterative schemes which are known as Mann, Ishikawa and Noor iterative schemes respectively is equivalent, for a special type of operator named Zamfirescu operator defined in a closed, convex subset of a Banach space. In [17] Murat Ozdemir and Sezgin Akbulut studied the equivalence of these three iterative schemes for the class of Lipschitzian operators, whereas we have studied for Zamfirescu operator. Our main results extend the results of Murat Ozdemir and Sezgin Akbulut [17] and the results of Rhoades and Soltuz [1-2].

KEYWORDS

Fixed point, Banach space, Mann, Ishikawa and Noor iterative schemes, Zamfirescu operator, $T$-stability.

1. INTRODUCTION

The first theorem on fixed point was established by Polish Mathematician Stefan Banach [7] in 1922. This theorem is known as Banach fixed point theorem or Contraction mapping theorem. Banach fixed point theorem has been applied to many different areas. For example, it can be used to prove the existence and uniqueness of solutions of certain differential equations. Another application of Banach fixed point theorem is it can be used to prove Implicit function theorem. The Mann iterative scheme, known as one-step iterative scheme [12], invented in 1953, was used to prove the convergence of the sequence to a fixed point of many valued mapping for which the Banach fixed point theorem [7] failed. Later, in 1974 Ishikawa [10] devised a new iteration scheme known as two-step iterative scheme to establish the convergence of Lipschitzian pseudocontractive map when Mann iteration scheme failed to converge. M.A. Noor [12-13] introduced and analyzed three-step iterative scheme to study the approximate solutions of variational inclusions (inequalities) in Hilbert spaces by using the techniques of updating the solution and the auxiliary principle. B. Xu and M.A. Noor [15] studied the convergence of Noor iterative scheme to
fixed point of an asymptotically nonexpensive self map defined in a closed, bounded and convex subset of a uniformly convex Banach space. A bulk of literature now exist around the theme of establishing the convergence of the Mann iteration for certain classes of mapping and then showing that the Ishikawa and Noor iterations also converges. In fact, proving the convergence of Ishikawa and Noor iterations the convergence of the corresponding Mann iteration can be obtained. Indeed, in many cases, if Mann iterative sequence for mapping $T$ converges then, Ishikawa and Noor iterative sequences also converge for that mapping. But this cannot be proved in general. In the light of this fact, recently, in a paper Rhoades and Soltuz [1-3] proved that Mann and Ishikawa iteration schemes are equivalent for several classes of mappings such as Lipschitzian, strongly pseudocontractive, strongly hemicontractive, strongly accretive, strongly successively pseudocontractive, strongly successively hemicontractive mapping and Krishna Kumar in [11] proved that Mann and Ishikawa schemes are equivalent for the class of uniformly pseudocontractive operators. In a paper, Murat Ozdemir and Sezgin Akbulut[17] proved that Mann, Ishikawa and Noor iterative schemes are equivalent for the class of Lipschitzian operators. In our present paper, we show that the convergence of Noor iterative scheme is equivalent to the convergence of Mann and Ishikawa iterative schemes for Zamfirescu operator.

The main purpose of this paper is to extend the results of Murat Ozdemir and Sezgin Akbulut[17] and the results of Rhoades and Soltuz [1, 2] for the class of Zamfirescu operator in Banach space for the purpose of proving the equivalence of iterative sequence generated by Mann, Ishikawa and Noor iterative schemes.

2. PRELIMINARIES

In 1972, T. Zamfirescu [18] obtained a very interesting fixed point theorem for the convergence of Picard iterative scheme, which is stated as follows:

**Theorem 2.1** ([Theorem Z, see [18]])

Let $X$ be a Banach space and $T : X \rightarrow X$ be a map for which there exist the real numbers $a, b$ and $c$ satisfying $0 < a < 1, 0 < b, c < 1/2$ such that for each pair $x, y$ in $X$ at least one of the following is true:

$(z_1) \quad \|Tx - Ty\| \leq a\|x - y\|;
(z_2) \quad \|Tx - Ty\| \leq b(\|x - Tx\| + \|y - Ty\|);
(z_3) \quad \|Tx - Ty\| \leq c(\|x - Ty\| + \|y - Tx\|).

Then $T$ have a unique fixed point $p$ and the Picard iterative scheme $\{p_n\}_{n=0}^{\infty}$ defined by $p_{n+1} = Tp_n$, $n = 0, 1, 2, \ldots$ converge to $p$ for any $p_0 \in X$.

**Definition 2.2**

Let $X$ be a Banach space. Then the operator $T : X \rightarrow X$ is called **Zamfirescu operator** if it satisfies one of the conditions $(z_1)$, $(z_2)$ and $(z_3)$. The class of Zamfirescu operators $T$ is one of the most studied class of quasi-contractive type operators, for which all important fixed point iteration schemes, i.e., Picard [51], Mann [54] and Ishikawa [41] iterations, are known to converge to the unique fixed point of $T$. Zamfirescu showed in [47] that an operator satisfying condition $Z$ has a unique fixed point that can be approximated using the Picard iteration. Later, B.E. Rhoades [3] proved that the Mann and Ishikawa iterations can also be used to approximate fixed points of Zamfirescu operator. The class of operators satisfying condition $Z$ is independent, see B.E. Rhoades [3], of the class of strictly (strongly) pseudocontractive operators, extensively studied by several authors in the last years. The set of fixed points of the operator $T$ is denoted by $F(T) = \{ p \in X :Tp = p \}$. 

**Definition 2.3**
Let $X$ be a Banach space, $B$ be a nonempty, convex subset of $X$ and $T : B \rightarrow B$ be an operator satisfying one of the conditions ($z_1$), ($z_2$) and ($z_3$) i.e., $T$ be a Zamfirescu operator.

Let $u_0, x_0, p_0 \in B$ be three arbitrary fixed points. Then the definitions of three iterative schemes are as follows:

The **Mann (one-step) iterative scheme** [16]

$\{u_n\}_{n=0}^{\infty}$ is defined by

$$u_{n+1} = (1-a_n)u_n + a_nTu_n, \quad n = 0,1,2,... \quad (i)$$

where the sequence $\{a_n\}_{n=0}^{\infty} \subset [0,1]$ is convergent, such that

$$\lim_{n \to \infty} a_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} a_n = \infty \quad (i.a)$$

The **Ishikawa (two-step) iterative scheme** [10]

$\{p_n\}_{n=0}^{\infty}$ is defined by

$$\begin{align*}
\{ \begin{array}{l}
x_{n+1} = (1-a_n)x_n + a_nTy_n \\
y_{n+1} = (1-b_n)y_n + b_nTx_n,
\end{array} \quad n = 0,1,2,... \quad (ii)\end{align*}$$

where the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty} \subset [0,1]$ are convergent, such that

$$\lim_{n \to \infty} a_n = 0, \quad \lim_{n \to \infty} b_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} a_n = \infty \quad (i.a)$$

The **Noor (three-step) iterative scheme** [12]

$\{p_n\}_{n=0}^{\infty}$ is defined by

$$\begin{align*}
p_{n+1} &= (1-a_n)p_n + a_nTq_n \\
qu_n &= (1-b_n)u_n + b_nTr_n \\
r_n &= (1-c_n)r_n + c_nTp_n, \quad n = 0,1,2,... \quad (iii)
\end{align*}$$

where the sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty} \subset [0,1]$ are convergent, such that

$$\lim_{n \to \infty} a_n = 0, \quad \lim_{n \to \infty} b_n = 0, \quad \lim_{n \to \infty} c_n = 0$$

and $\sum_{n=1}^{\infty} a_n = \infty \quad (iii.a)$

**Definition 2.4**

Let $F(T) = \{ p \in X : T(p) = p \},$ $p \in F(T).$ Consider

$$\eta_n = \| u_{n+1} - (1-a_n)u_n - a_n Tu_n \| \quad (iv.a)$$

$$\mu_n = \| p_{n+1} - (1-a_n)p_n - a_n Tq_n \| \quad (iv.b)$$

$$\xi_n = \| x_{n+1} - (1-a_n)x_n - a_n T\bar{x}_n \| \quad (iv.c)$$

If $\lim_{n \to \infty} \eta_n = 0, \lim_{n \to \infty} \mu_n = 0$ and $\lim_{n \to \infty} \xi_n = 0,$

then the iterative schemes (i), (ii) and (iii) respectively are said to be $T$-stable.

**Lemma 2.5**

Let $X$ be a Banach space, $B$ be a nonempty, convex subset of $X$ and $T : B \rightarrow B$ be a Zamfirescu operator. If the Mann (respectively Ishikawa and Noor) iterative scheme converges, then $\lim_{n \to \infty} \eta_n = 0$ (respectively $\lim_{n \to \infty} \mu_n = 0$ and $\lim_{n \to \infty} \xi_n = 0$).

In [8] V. Berinde and in [6] B. E. Rhoades proved the following convergence theorems in Banach spaces, for Mann and Ishikawa iterations associated to Zamfirescu operator.

**Theorem 2.6** (see [8])

Let $X$ be an arbitrary Banach space, $B$ be a nonempty closed convex subset of $X$ and $T : B \rightarrow B$ be an operator satisfying condition $Z$.

Let $\{u_n\}_{n=0}^{\infty}$ be the Mann iteration defined by (i) and (i.a). Then $\{u_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of $T$.

**Theorem 2.7** (see [6])

Let $X$ be an arbitrary Banach space, $B$ be a nonempty closed convex subset of $X$ and $T : B \rightarrow B$ be an operator satisfying condition $Z$. 

18
Let \( \{x_n\}_{n=0}^{\infty} \) be the Ishikawa iteration defined by (ii) and (ii.a). Then \( \{x_n\}_{n=0}^{\infty} \) converges strongly to the fixed point of \( T \).

3. MAIN RESULTS

Now, we state and prove our main results.

**Theorem 3.1**

Let \( X \) be an arbitrary Banach space, \( B \) be a nonempty closed convex subset of \( X \) and \( T : B \to B \) be an operator satisfying condition \( Z \) i.e., \( T : B \to B \) be a Zamfirescu operator. Let \( p \in F(T) \) be a fixed point of \( T \) where \( F(T) \) denotes the set of fixed points of \( T \).

Let \( \{u_n\}_{n=0}^{\infty} \) be the Mann iteration defined by (i) and (i.a), \( \{x_n\}_{n=0}^{\infty} \) be the Ishikawa iteration defined by (ii) and (ii.a) and \( \{p_n\}_{n=0}^{\infty} \) be the Noor iteration defined by (iii) and (iii.a). Then the following assertions are equivalent:

(a) The Mann iterative scheme (i) and (i.a) is converges to \( p \in F(T) \);

(b) The Ishikawa iterative scheme (ii) and (ii.a) is converges to \( p \in F(T) \);

(c) The Noor iterative scheme (iii) and (iii.a) is converges to \( p \in F(T) \).

**Proof**

We prove our theorem in the following three steps: step-1: \( (a) \iff (b) \), step-2: \( (b) \iff (c) \) and step-3: \( (c) \iff (a) \).

**Step-1:** In this step we first prove that \( (b) \implies (a) \). Suppose that the Ishikawa iteration scheme converges to \( p \). Then it is clear that this \( p \) is a fixed point of \( T \), i.e., \( Tp = p \). By setting, \( b_n = 0 \ \forall \ n \in \mathbb{N} \) (set of all natural numbers) in (ii) we obtain the convergence of Mann iterative scheme (i).

Conversely, we prove that \( (a) \implies (b) \) i.e. the convergence of Mann iterative scheme to the fixed point \( p \) implies the convergence of Ishikawa iterative scheme to the fixed point \( p \).

Now, by Theorem 2.1(Z) we know that \( T \) has a unique fixed point in \( B \), say \( p \). Consider \( x, y \in B \).

Since \( T \) is a Zamfirescu operator, therefore, at least one of the conditions \( (z_1) \), \( (z_2) \) and \( (z_3) \) is satisfied by \( T \). If \( (z_2) \) holds, then

\[
\|Tx - Ty\| \leq b\|x - Tx\| + \|y - Ty\|
\]

\[
\leq b\|x - Tx\| + (1 - b)\|y - x\|
\]

\[
+ \|x - Tx\| + \|Ty - Ty\|
\]

\[
\iff \|Tx - Ty\| \leq \frac{b}{1 - b}\|x - y\|
\]

(1)

If \( (z_3) \) holds, then similarly we obtain

\[
\|Tx - Ty\| \leq \frac{c}{1 - c}\|x - y\|
\]

(2)

Let us denote

\[
\lambda = \max\left\{a, \frac{b}{1 - b}, \frac{c}{1 - c}\right\}
\]

(3)

Then we have, \( 0 \leq \lambda < 1 \) and in view of \( (z_1) \), (1) and (2) we get the following inequality

\[
\|Tx - Ty\| \leq \lambda\|x - y\| + 2\lambda\|x - Tx\|
\]

(4)

holds \( \forall \ x, y \in B \).
Since \( \{x_n\}_{n=0}^{\infty} \) be the Ishikawa iterative scheme defined by (ii) and \( \{u_n\}_{n=0}^{\infty} \) be the Mann iterative scheme defined by (i), therefore, we get
\[
\|x_{n+1} - u_{n+1}\|
= \|(1-a_n)x_n + a_nT_{y_n} - (1-a_n)u_n + a_nT_{u_n}\|
= \|(1-a_n)(x_n - u_n) + a_n(T_{y_n} - T_{u_n})\|
\leq (1-a_n)\|x_n - u_n\| + a_n\|T_{y_n} - T_{u_n}\|
\quad (5)
\]

Now, according to the supposition Mann iterative scheme \( \{u_n\}_{n=0}^{\infty} \) converge to the fixed point \( p \). i.e., \( \lim_{n \to \infty} u_n = p \). This implies that
\[
\lim_{n \to \infty}[T_{u_n} - u_n] = Tp - p = 0.
\]
\[
\Rightarrow \lim_{n \to \infty} T_{u_n} = \lim_{n \to \infty} u_n = p
\quad (6)
\]
\[
\Rightarrow \lim_{n \to \infty} T_{u_n} = p
\quad (7)
\]
Again, by Mann iterative scheme (i) we have
\[
u_{n+1} = (1-a_n)u_n + a_nT_{u_n}
\]
\[
\Rightarrow u_{n+1} - u_n = a_n[T_{u_n} - u_n]
\quad (8)
\]
From, (6) and (8) we can write,
\[
\lim_{n \to \infty}[u_{n+1} - u_n] = 0 \Rightarrow \lim_{n \to \infty} u_{n+1} = p
\quad (9)
\]
Now, combining (5), (6) and (7) we obtain,
\[
\|x_{n+1} - u_{n+1}\| \leq (1-a_n)\|x_n - p\| + a_n\|y_n - p\|
\quad (10)
\]
By setting, \( x = p \) & \( y = y_n \) in (4) we get,
\[
\|T_{y_n} - p\| \leq \|y_n - p\|
\quad (11)
\]
Where \( \lambda \) is given by (3).
We have,
\[
\|y_n - p\| = \|(1-b_n)x_n + b_nT_{x_n} - (1-b_n-b_n)p\|
= \|(1-b_n)(x_n - p) + b_n(T_{x_n} - p)\|
\leq (1-b_n)\|x_n - p\| + b_n\|T_{x_n} - p\|
\quad (12)
\]
Again, by setting \( x = p \) & \( y = x_n \) in (4) we get,
\[
\|T_x_n - p\| \leq \|x_n - p\|
\quad (13)
\]
From (12) and (13) we obtain,
\[
\|y_n - p\| \leq (1-b_n)\|x_n - p\|
\quad (14)
\]
Now, combining (10), (11) and (14) we obtain,
\[
\|x_{n+1} - u_{n+1}\|
\leq (1-a_n)\|x_n - p\| + \lambda a_n[(1-b_n)\|x_n - p\|
\quad + b_n\|x_n - p\|]
\leq [1 - (1-\lambda)a_n(1-\lambda b_n)]\|x_n - p\|
\quad (15)
\]
By (15), we inductively obtain,
\[
\|x_{n+1} - u_{n+1}\| \leq \prod_{k=0}^{n}[1 - (1-\lambda)^2a_k]\|x_0 - p\|
\quad (16)
\]
where, \( n = 0, 1, 2, \ldots \).
Using the fact that
\[
0 \leq \lambda < 1, a_k \in [0, 1] & \sum_{n=0}^{\infty} a_n = \infty
\]
we obtain,
\[
\lim_{n \to \infty}\prod_{k=0}^{n}[1 - (1-\lambda)^2a_k] = 0
\quad (17)
\]
Comparing, (16) and (17) we get,
\[
\lim_{n \to \infty}\|x_{n+1} - u_{n+1}\| = 0
\]
\[
\Rightarrow \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} u_{n+1} \Rightarrow \lim_{n \to \infty} x_{n+1} = p
\quad (18)
\]
From, which, we can say that, \( \{x_n\}_{n=0}^{\infty} \) converges to the fixed point \( p \), i.e., the Ishikawa iterative scheme (ii) converges to \( p \). This completes the step-1 of our proof.

**Step-2: In this step we first prove that (c) \( \Rightarrow (b) \). Suppose that the Noor iteration scheme (iii) converges to \( p \). Then it is clear that this \( p \) is a fixed point of \( T \), i.e., \( Tp = p \). By setting, \( c_n = 0 \ \forall \ n \in \mathbb{N} \) (set of all natural
numbers), in (iii) we obtain the convergence of Ishikawa iterative scheme (ii).

Conversely, we prove that \( b \Rightarrow c \) i.e., the convergence of Ishikawa iterative scheme to the fixed point \( p \) implies the convergence of Noor iterative scheme to the fixed point \( p \).

Since \( \{x_n\}_{n=0}^{\infty} \) be the Ishikawa iterative scheme defined by (ii) and \( \{p_n\}_{n=0}^{\infty} \) be the Noor iterative scheme defined by (iii), therefore, we get

\[
\|p_{n+1} - x_{n+1}\| = \|(1 - a_n)p_n + a_n T q_n - (1 - a_n)x_n - a_n T y_n\|
\leq (1 - a_n)\|p_n - x_n\| + a_n \|T q_n - T y_n\|
\leq (1 - a_n)\|p_n - x_n\| + a_n \|a_n (1 - b_n)\|\|p_n - x_n\|
+ a a b_n \|T r_n - T x_n\|
= \|(1 - a_n) + a_n a(1 - b_n)\|\|p_n - x_n\|
+ a a b_n \|T r_n - T x_n\|
\]

Now, according to the supposition, Ishikawa iterative scheme \( \{x_n\}_{n=0}^{\infty} \) converge to the fixed point \( p \), i.e., \( \lim_{n \to \infty} x_n = p \).

\[
\lim_{n \to \infty} [T x_n - x_n] = T p - p = 0
\Rightarrow \lim_{n \to \infty} T x_n = \lim_{n \to \infty} x_n = p
\Rightarrow \lim_{n \to \infty} T x_n = p
\]

Again, by Ishikawa iterative scheme we have,

\[
x_{n+1} = (1 - a_n)x_n + a_n T [(1 - b_n)x_n + b_n T x_n]
\Rightarrow x_{n+1} - x_n = -a_n x_n
+ a_n T [x_n - b_n (x_n - T x_n)]
\Rightarrow \lim_{n \to \infty} (x_{n+1} - x_n) = 0
\Rightarrow \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} x_n = p
\]

Combining (18), (19) and (20) we get,

\[
\|p_{n+1} - x_{n+1}\| \leq \{(1 - a_n) + a_n a (1 - b_n)\|p_n - p\|
+ a a b_n \|T r_n - T p\|
\]

Now, by setting \( x = p \) & \( y = r_n \) in (4) we get,

\[
\|T r_n - p\| \leq \lambda \|r_n - p\|
\]

where, \( \lambda \) is given by (3).

We have,

\[
\|r_n - p\| = \|(1 - c_n)p_n + c_n T p_n - (1 - c_n + c_n)p\|
= \|(1 - c_n)(p_n - p) + c_n T p_n - p\|
\leq (1 - c_n)\|p_n - p\| + c_n \|T p_n - p\|
\]

By setting \( x = p \) & \( y = p_n \) in (4) we obtain,

\[
\|T p_n - p\| \leq \lambda \|p_n - p\|
\]

where, \( \lambda \) is given by (3).

From, (24) and (25) we get,

\[
\|r_n - p\| \leq (1 - c_n)\|p_n - p\| + c_n \lambda \|p_n - p\|
\]

Combining, (22), (23) and (26) we get,

\[
\|p_{n+1} - x_{n+1}\| \leq \{(1 - a_n) + a_n a (1 - b_n)\|p_n - p\|
+ \lambda a a b_n (1 - c_n)\|p_n - p\|
+ \lambda^2 a a b_n c_n \|p_n - p\|
= \{1 - (1 - a + (1 - \lambda) a b_n (1 + \lambda c_n)) a_n\}\|p_n - p\|
\]
$$= [1 - (1 - a)a_n - (1 - \lambda)aa_n b_n (1 + \lambda c_n)]$$

$$\| p_n - p \| \leq [1 - (1 - a)a_n] \| p_n - p \|$$

By (27) we inductively obtain,

$$\| p_{n+1} - x_{n+1} \| \leq \sum_{k=0}^{n} [1 - (1 - a)a_k] \| p_0 - p \|.$$  \hspace{1cm} (27.a)

where, \( n = 0, 1, 2, \ldots \)

Now, using the fact that \( 0 < a < 1, \ a_k \in [0, 1] \) \& \( \sum_{n=0}^{\infty} a_n = \infty \),

we obtain, \( \lim_{n \to \infty} \sum_{k=0}^{n} [1 - (1 - a)a_k] = 0 \)  \hspace{1cm} (28)

Comparing, (27.a) and (28) we get,

$$\lim_{n \to \infty} \| p_{n+1} - x_{n+1} \| = 0$$

$$\Rightarrow \lim_{n \to \infty} p_{n+1} = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} x_n = p.$$  \hspace{1cm} (29)

$$\Rightarrow \lim_{n \to \infty} p_{n+1} = p$$

From, which we can say that, \( \{ p_n \}_{n=0}^{\infty} \) converges to the fixed point \( p \). i.e., the Noor iterative scheme converges (iii) to \( p \). This completes the step-2 of our proof.

**Step-3:** In this step we first prove that \( (c) \Rightarrow (a) \). Suppose that the Noor iterative scheme (iii) converges to \( p \). Then it is clear that this \( p \) is a fixed point of \( T \). i.e., \( Tp = p \). By setting, \( b_n = 0 \) \& \( c_n = 0 \ \forall \ n \in \mathbb{N} \) (set of all natural numbers), in (iii) we obtain the convergence of Mann iterative scheme (i).

Conversely, we prove that \( (a) \Rightarrow (c) \). i.e., the convergence of Mann iterative scheme (i) to the fixed point \( p \) implies the convergence of Noor iterative scheme (iii) to the fixed point \( p \). Since \( \{ u_n \}_{n=0}^{\infty} \) be the Mann iterative scheme defined by (i) and \( \{ p_n \}_{n=0}^{\infty} \) be the Noor iterative scheme defined by (iii), therefore, we get

$$\| p_{n+1} - u_{n+1} \| = \| (1 - a_n) p_n + a_n Tq_n - (1 - a_n) u_n - a_n Tu_n \|$$

$$\leq (1 - a_n) \| p_n - u_n \| + a_n \| Tq_n - Tu_n \|$$  \hspace{1cm} (30)

Now, combining (6), (7) and (29) we obtain,

$$\| p_{n+1} - u_{n+1} \| \leq (1 - a_n) \| p_n - u_n \| + a_n \| Tq_n - Tu_n \|$$  \hspace{1cm} (31)

We have,

$$\| q_n - p \|$$

$$= \| (1 - b_n) p_n + b_n Tq_n - (1 - b_n + b_n) p \|$$  \hspace{1cm} (32)

$$= \| (1 - b_n) (p_n - p) + b_n Tq_n - p \|$$

$$\leq (1 - b_n) \| p_n - p \| + b_n \| Tq_n - p \|$$

Again, by setting \( x = p \) \& \( y = q_n \) in (4) we get,

$$\| Tp_n - p \| \leq \lambda \| p_n - p \|$$  \hspace{1cm} (33)

We have,

$$\| p_n - p \|$$

$$= \| (1 - c_n) p_n + c_n Tp_n - (1 - c_n + c_n) p \|$$  \hspace{1cm} (34)

$$= \| (1 - c_n) (p_n - p) + c_n Tp_n - p \|$$

$$\leq (1 - c_n) \| p_n - p \| + c_n \| Tp_n - p \|$$

Again, by setting \( x = p \) \& \( y = p_n \) in (4) we get,

$$\| Tp_n - p \| \leq \lambda \| p_n - p \|$$  \hspace{1cm} (35)

From (34) and (35) we obtain,

$$\| p_n - p \| \leq (1 - c_n) \| p_n - p \| + c_n \lambda \| p_n - p \|$$

From (33) and (36) we get,
be the Noor iteration defined by 
\[
T_n = \lambda [(1 - c_n) p_n - p] + c_n \lambda [p_n - p]
\]
From (32) and (37) we get,
\[
\| q_n - p \| \\
\leq (1 - b_n) \| p_n - p \| + b_n \lambda [(1 - c_n) p_n - p] + c_n \lambda [p_n - p] \]
Combining (30) and (39) we get,
\[
\| p_{n+1} - u_{n+1} \| \\
\leq (1 - a_n) \| p_n - p \| + a_n \lambda [(1 - b_n) p_n - p] + b_n \lambda [(1 - c_n) p_n - p] + c_n \lambda [p_n - p] \]
\[= [1 - a_n (1 - \lambda)] - a_n b_n c_n \lambda (1 - \lambda) - a_n b_n c_n \lambda^2 (1 - \lambda) \| p_n - p \| \]
\[= [1 - a_n (1 - \lambda)] (1 + b_n \lambda) + b_n c_n \lambda^2 \| p_n - p \| \]
\[\leq [1 - a_n (1 - \lambda)] \| p_n - p \| \]
By (40), we inductively obtain,
\[
\| p_{n+1} - u_{n+1} \| \leq \prod_{k=0}^{n} [1 - (1 - \lambda) a_k] \| p_0 - p \|,
\]
where, \( n = 0, 1, 2, \ldots \) (41)
Using the fact that
\( 0 \leq \lambda < 1, a_k \in [0, 1], \sum_{n=0}^{\infty} a_n = \infty \)
we obtain, \( \lim_{n \to \infty} \prod_{k=0}^{n} [1 - (1 - \lambda) a_k] = 0 \) (42)
Comparing, (41) and (42) we get,
\[
\lim_{n \to \infty} \| p_{n+1} - u_{n+1} \| = 0
\]
\( \Rightarrow \lim_{n \to \infty} p_{n+1} = \lim_{n \to \infty} u_{n+1} \) \[\text{By equation (9)}\]
\( \Rightarrow \lim_{n \to \infty} p_{n+1} = p \)
From, which we can say that, \( \{p_n\}_{n=0}^{\infty} \) converges to the fixed point \( p \). i.e., the Noor iterative scheme converges to \( p \). This completes the step-3 of our proof.

Now, we give an example, which proves numerically the trueness of our theorem 3.1.

Example 3.2

Let \( X = \mathbb{R} \) (set of all real numbers), \( B = [0, 2] \) and \( T : B \to B \) be a Zamfirescu operator defined by \( Tx = \frac{x + 1}{2} \).
\( T \) has a fixed point \( p = 1 \in B \). Now, let us choose the sequences \( \{a_n\}_{n=0}^{\infty} \), \( \{b_n\}_{n=0}^{\infty} \) and \( \{c_n\}_{n=0}^{\infty} \) such that
\( a_n = \frac{1}{n + 1} \), \( b_n = \frac{1}{n + 2} \) and \( c_n = \frac{1}{n + 3} \); respectively and \( x_0 = 0.1 \in B \). Then, all conditions of our theorem 3.1 are satisfied.

Theorem 3.3

Let \( X \) be a Banach space, \( B \) be a nonempty closed convex subset of \( X \) and \( T : B \to B \) be an operator satisfying condition \( Z \) i.e., \( T : B \to B \) be a Zamfirescu operator. Let \( p \in F(T) \) be a fixed point of \( T \) where \( F(T) \) denotes the set of fixed points of \( T \).
Let \( \{u_n\}_{n=0}^{\infty} \) be the Mann iteration defined by (i) and (i.a), \( \{x_n\}_{n=0}^{\infty} \) be the Ishikawa iteration defined by (ii) and (ii.a) and \( \{p_n\}_{n=0}^{\infty} \) be the Noor iteration defined by (iii) and (iii.a). Then the following assertions are equivalent:
(a) The Mann iterative scheme (i) and (i.a) is \( T \)-stable;
(b) The Ishikawa iterative scheme (ii) and (ii.a) is \( T \)-stable;
(c) The Noor iterative scheme (iii) and (iii.a) is \( T \)-stable.

**Proof**

We prove our theorem in the following three steps: step-1: (a) \( \iff \) (b) step-2: (b) \( \iff \) (c) and step-3: (a) \( \iff \) (c).

**Step-1:** From definition 2.3, (a) \( \iff \) (b) means that \( \lim_{n \to \infty} \eta_n = 0 \implies \lim_{n \to \infty} \mu_n = 0 \) and so, \( \lim_{n \to \infty} \mu_n = 0 \implies \lim_{n \to \infty} \eta_n = 0 \) is obvious by setting, \( b_n = 0 \ \forall \ n \in \mathbb{N} \) (set of all natural numbers) in (ii). Conversely, suppose that Mann iterative scheme (i) is \( T \)-stable. Using definition 2.3, we get

\[ \lim_{n \to \infty} \eta_n = 0 \implies \lim_{n \to \infty} u_n = p . \]

Thus, we get \( \lim_{n \to \infty} \eta_n = 0 \implies \lim_{n \to \infty} \mu_n = p . \) This completes the step-1 of our proof.

**Step-2:** From definition 2.3, (b) \( \iff \) (c) means that \( \lim_{n \to \infty} \mu_n = 0 \iff \lim_{n \to \infty} \xi_n = 0 \) and so, \( \lim_{n \to \infty} \xi_n = 0 \implies \lim_{n \to \infty} \mu_n = 0 \) is obvious by setting, \( c_n = 0 \ \forall \ n \in \mathbb{N} \) (set of all natural numbers) in (ii). Conversely, suppose that Ishikawa iterative scheme (vi) is \( T \)-stable. Using definition 2.3, we get \( \lim_{n \to \infty} \mu_n = 0 \implies \lim_{n \to \infty} x_n = p . \) Now, by theorem 3.1 we get, \( \lim_{n \to \infty} x_n = p \implies \lim_{n \to \infty} p_n = p . \) Using lemma 2.4 we have, \( \lim_{n \to \infty} \xi_n = 0 . \)

Thus, we get \( \lim_{n \to \infty} \mu_n = 0 \implies \lim_{n \to \infty} \xi_n = 0 . \) This completes the step-2 of our proof.

**Step-3:** From definition 2.3, (c) \( \iff \) (a) means that \( \lim_{n \to \infty} \xi_n = 0 \iff \lim_{n \to \infty} \eta_n = 0 \) and so, \( \lim_{n \to \infty} \xi_n = 0 \implies \lim_{n \to \infty} \eta_n = 0 \) is obvious by setting, \( b_n \) & \( c_n = 0 \ \forall \ n \in \mathbb{N} \) (set of all natural numbers), in (iii).

Conversely, suppose that Mann iterative scheme (i) is \( T \)-stable. Using definition 2.3, we get \( \lim_{n \to \infty} \eta_n = 0 \implies \lim_{n \to \infty} u_n = p . \) Now, by theorem 3.1 we get, \( \lim_{n \to \infty} u_n = p \implies \lim_{n \to \infty} p_n = p . \) Using lemma 2.4 we have, \( \lim_{n \to \infty} \xi_n = 0 . \)

Thus, we get \( \lim_{n \to \infty} \eta_n = 0 \implies \lim_{n \to \infty} \xi_n = 0 . \) This completes the step-3 of our proof.

**4. CONCLUSION**

Our theorem 3.1, extended the results of Murat Ozdemir and Sezgin Akbulut [17] and the results of Rhoades and Soltuz [1, 2], because here we replaced the Lipschitzian operator by the Zamfirescu operator and the Zamfirescu operator is a more general operator compared to Lipschitzian operator. Finally, by our theorem 3.2, we have shown that the \( T \)-stabilities of one-step, two-step and three-step iterative schemes are also equivalent for Zamfirescu operator.

**REFERENCES**


